The number of Hecke eigenvalues of same signs

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THE NUMBER OF HECKE EIGENVALUES OF SAME SIGNS

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Abstract. We give the best possible lower bounds in order of magnitude for the number of positive and negative Hecke eigenvalues. This improves upon a recent work of Kohnen, Lau & Shparlinski. Also, we study an analogous problem for short intervals.

1. Introduction

Let $k \geqslant 2$ be an even integer and $N \geqslant 1$ be squarefree. Among all holomorphic cusp forms of weight k for the congruence subgroup $\Gamma_0(N)$, there are finitely many of them whose Fourier coefficients in the expansion at the cusp ∞,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m z > 0),$$

are the Hecke eigenvalues. Up to scalar multiples, these forms are the only simultaneous eigenfunctions of all Hecke operators. We call them the primitive forms, and write $\mathrm{H}_k^*(N)$ for the set of all primitive forms of weight k for $\Gamma_0(N)$. One central problem in modular form theory is to study the Hecke eigenvalues $\lambda_f(n)$. (We omit the factor $n^{(k-1)/2}$ to avoid its uneven amplifying effect.) Classically it is known that the arithmetical function $\lambda_f(n)$ is real multiplicative, and verifies Deligne's inequality

$$|\lambda_f(n)| \leqslant d(n) \tag{1.1}$$

for all $n \geqslant 1$, where $d(n)$ is the divisor function. Furthermore we have

$$\lambda_f(p^\nu) = \lambda_f(p)^\nu \qquad \text{and} \qquad \lambda_f(p) = \varepsilon_f(p)/\sqrt{p} \tag{1.2}$$

for all primes $p \mid N$ and integers $\nu \geqslant 1$, where $\varepsilon_f(p) \in \{\pm 1\}$. (See [5] and [10].) The distribution of the Hecke eigenvalues $\lambda_f(n)$ is delicate. The Lang-Trotter conjecture concerns the frequency of $\lambda_f(p)$ taking a value in the admissible range where p runs over primes. This conjecture is still open but there are progress made on itself or the pertinent questions, for instance, [6], [18], [16], [17], [2], [4], [15], etc. In this regard, various techniques and tools are applied, such as ℓ-adic representations, Chebotarev density theorem, sieve-theoretic arguments, Rankin-Selberg L-functions and the method of $\mathscr{B}$-free numbers. In [15], Kowalski, Robert & Wu investigated the nonvanishing problem and gave the sharpest upper estimate to-date on the gaps between consecutive nonzero Hecke eigenvalues. Another wide belief is Sato-Tate's conjecture, asserting that $\lambda_f(p)$'s are equidistributed on $[-2, 2]$ with respect to the Sato-Tate measure.

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In this paper, we are concerned with the Hecke eigenvalues of the same sign. Kohnen, Lau & Shparlinski [14, Theorem 1] proved

$$\mathscr{N}_f^{\pm}(x) := \sum_{\substack{n\leqslant x,\,(n,N)=1\\ \lambda_f(n)\gtrless 0}} 1 \gg_f \frac{x}{(\log x)^{17}} \tag{1.3}$$

for $x \geqslant x_0(f)$. † Very recently Wu [21, Corollary] improved this result by reducing the exponent 17 to $1-1/\sqrt{3}$, as a simple application of his estimates on power sums of Hecke eigenvalues. The exponent $1-1/\sqrt{3}$ can be improved to $2-16/(3\pi)$ if one assumes Sato-Tate's conjecture.

Our first result is to remove the logarithmic factor by the $\mathscr{B}$-free number method, which is the best possible in order of magnitude.

Theorem 1. *Let $f \in \mathrm{H}_k^*(N)$. Then there is a constant x_0 such that the inequality*

$$\mathscr{N}_f^{\pm}(x) \gg_f x \tag{1.4}$$

holds for all $x \geqslant x_0$.

Remarks. 1. It is clear from the proof that our method gives the stronger result

$$\sum_{\substack{n\leqslant x,\,(n,N)=1\\ n\,\text{squarefree},\,\lambda_f(n)\gtrless 0}} 1 \gg_f x$$

for every $x \geqslant x_0(f)$.

2. The method is robust and applies to, for example, modular forms of half-integral weight. We return to this problem in another occasion.

By coupling (1.3) with Alkan & Zaharescu's result in [1, Theorem 1], it is shown in [14, Theorem 2] (see also [13, Theorem 3.4]) that there are absolute constants $\eta < 1$ and $A > 0$ such that for any $f \in \mathrm{H}_k^*(N)$ the inequality

$$\mathscr{N}_f^{\pm}(x + x^{\eta}) - \mathscr{N}_f^{\pm}(x) > 0 \tag{1.5}$$

holds for $x \geqslant (kN)^A$, but no explicit value of η is evaluated. Apparently it is interesting and important to know how small η can be, in order for a better understanding of the local behaviour. A direct consequence of (1.5) is that $\lambda_f(n)$ has a sign-change in a short interval $[x, x+x^{\eta}]$ for all sufficiently large x. The sign-change problem was explored in [11], [14], [21] on different aspects. Here we prove that there are plenty of eigenvalues of the same signs in intervals of length about $x^{1/2}$. More precisely, we have the following.

Theorem 2. *Let $f \in \mathrm{H}_k^*(N)$. There is an absolute constant $C > 0$ such that for any $\varepsilon > 0$ and all sufficiently large $x \geqslant N^2 x_0(k)$, we have*

$$\mathscr{N}_f^{\pm}(x + C_N x^{1/2}) - \mathscr{N}_f^{\pm}(x) \gg_{\varepsilon} (Nx)^{1/4-\varepsilon}, \tag{1.6}$$

where

$$C_N := CN^{1/2}\Psi(N)^3, \qquad \Psi(N) := \sum_{d\mid N} d^{-1/2}\log(2d)$$

and $x_0(k)$ is a suitably large constant depending on k and the implied constant in $\gg_{\varepsilon}$ depends only on ε.

†It is worthy to indicate that they gave explicit values for the implied constant in $\gg$ and $x_0(f)$.

The result in Theorem 2 is uniform in the level N, and its method of proof is based on Heath-Brown & Tsang [8]. The exponent of $\Psi(N)$ in C_N can be easily reduced to any number bigger than 3/2, which however may not be essential as $\Psi(N)$ is already very small - $\log \Psi(N) = o(\sqrt{\log N})$. The range of $x \geqslant N^2 x_0(k)$ can also be refined to $x \geqslant N^{1+\varepsilon} k^A$ for some constant $A > 0$, but we save our effort.

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2. Proof of Theorem 1

Let p' be the least prime such that $p' \nmid N$ and $\lambda_f(p') < 0$. [‡] Introduce the set

$$\begin{aligned}\mathscr{B} &= \{p : \lambda_f(p) = 0\} \cup \{p : p \mid N\} \cup \{p'\} \cup \{p^2 : p \nmid p'N \text{ and } \lambda_f(p) \neq 0\} \\ &= \{b_i\}_{i \geqslant 1} \quad \text{(with increasing order).}\end{aligned}$$

By virtue of Serre's estimate [18, (181)]:

$$|\{p \leq x \,:\, \lambda_f(p) = 0\}| \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}}$$

for $x \geq 2$ and any $\delta < \frac{1}{2}$, we infer that

$$\sum_{i \geqslant 1} 1/b_i < \infty \qquad \text{and} \qquad (b_i, b_j) = 1 \quad (i \neq j).$$

Let $\mathscr{A} := \{a_i\}_{i \geqslant 1}$ (with increasing order) be the sequence of all $\mathscr{B}$-free numbers, i.e. the integers indivisible by any element in $\mathscr{B}$. According to [7], $\mathscr{A}$ is of positive density

$$\lim_{x \to \infty} \frac{|\mathscr{A} \cap [1, x]|}{x} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{b_i}\right) > 0. \tag{2.1}$$

From the definition of $\mathscr{B}$ and the multiplicativity of $\lambda_f(n)$, we have $\lambda_f(a) \neq 0$ for all $a \in \mathscr{A}$. Then we partition

$$\mathscr{A} = \mathscr{A}^+ \cup \mathscr{A}^-,$$

where

$$\mathscr{A}^{\pm} := \big\{a_i \in \mathscr{A} : \lambda_f(a_i) \gtrless 0\big\}.$$

Without control on the sizes of $\mathscr{A}^{\pm}$, we construct a set from $\mathscr{A}^+ \cup \mathscr{A}^-$ such that the sign of $\lambda_f(a)$ is switched on the counterpart. Consider

$$\mathscr{N}^{\pm} := \mathscr{A}^{\pm} \cup \{a_i p' : a_i \in \mathscr{A}^{\mp}\}.$$

[‡]According to [11], we have $p' \ll (k^2 N)^{29/60}$.

Clearly $\lambda_f(a) \gtrless 0$ and $(a, N) = 1$ for all $a \in \mathscr{N}^{\pm}$ and

$$\mathscr{N}_f^{\pm}(x) \geqslant \left|\mathscr{N}^{\pm} \cap [1, x]\right| \geqslant \left|\mathscr{A} \cap [1, x/p']\right|$$

for all $x \geqslant 1$. The desired result follows with the inequality (2.1).

3. Proof of Theorem 2

The method of proof is based on the investigation of

$$S_f^*(x) := \sum_{n \leqslant x,\, (n,N)=1} \lambda_f(n).$$

Since the L-function associated to f is belonged to the Selberg class and of degree 2, we apply the standard complex analysis to derive truncated Voronoi formulas for $S_f^*(x)$.

Lemma 3.1. *Let $f \in \mathrm{H}_k^*(N)$. Then for any $A > 0$ and $\varepsilon > 0$, we have*

$$\begin{aligned} S_f^*(x) &= \frac{\eta_f}{\pi\sqrt{2}}(Nx)^{1/4} \sum_{d|N} \frac{(-1)^{\omega(d)}\lambda_f(d)}{d^{1/4}} \sum_{n \leqslant M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{dN}} - \frac{\pi}{4}\right) \\ &\quad + O\left(N^{1/2}\left\{1 + \left(\frac{x}{M}\right)^{1/2} + \left(\frac{N}{x}\right)^{1/4}\right\}(Nx)^{\varepsilon}\right) \end{aligned} \tag{3.1}$$

uniformly for $1 \leqslant M \leqslant x^A$ and $x \geqslant N^{1+\varepsilon}$, where $\eta_f = \pm 1$ depends on f and the implied O-constant depends on A, ε and k only. The function $\omega(d)$ counts the number of all distinct prime factors of d.

Remark. The case $N = 1$ and $A = 1$ of (3.1) is covered in [12, Theorem 1.1] with $h = k = 1$ therein. Our proof follows closely Section 3.2 of [9], and we first evaluate the case without the constraint $(n, N) = 1$: for any $A > 0$ and $\varepsilon > 0$, we have uniformly in $1 \leqslant M \leqslant x^A$,

$$\begin{aligned} S_f(x) &:= \sum_{n \leqslant x} \lambda_f(n) \\ &= \frac{\eta_f (Nx)^{1/4}}{\pi\sqrt{2}} \sum_{n \leqslant M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) \\ &\quad + O\left(N^{1/2}\left\{1 + \left(\frac{x}{M}\right)^{1/2} + \left(\frac{N}{x}\right)^{1/4}\right\}(Nx)^{\varepsilon}\right). \end{aligned} \tag{3.2}$$

Proof. As usual, denote by $\mu(N)$ the Möbius function. (3.1) follows from (3.2) because

$$\begin{aligned} S_f^*(x) &= \sum_{d|N} \mu(d) \sum_{n \leqslant x/d} \lambda_f(dn) \\ &= \sum_{d|N} (-1)^{\omega(d)} \lambda_f(d) \sum_{n \leqslant x/d} \lambda_f(n) \end{aligned} \tag{3.3}$$

by the multiplicativity of $\lambda_f(n)$ and the first equality in (1.2). Note that $x/d \geqslant x^{\varepsilon/(1+\varepsilon)}$ when $x \geqslant N^{1+\varepsilon}$ and $d|N$, we can keep the same range of M for all inner sums over n by selecting a suitable A. Inserting (3.2) into (3.3), the main term of (3.1) comes up immediately. The effect of summing the O-terms over $d|N$ is negligible in light of the second formula in (1.2), and hence the result.

To prove (3.2), we consider $M \in \mathbb{N}$ without loss of generality. As usual write

$$L(s,f) := \sum_{n\geqslant 1} \lambda_f(n) n^{-s} \qquad (\Re e\, s > 1).$$

Let $\kappa := 1+\varepsilon$ and $T > 1$ be a parameter, chosen as

$$T^2 = \frac{4\pi^2(M+\frac{1}{2})x}{N}. \tag{3.4}$$

By the truncated Perron formula (see [20, Corollary II.2.4] with the choice of $\sigma_a = 1$, $\alpha = 2$ and $B(n) = C_\varepsilon n^\varepsilon$), we have

$$S_f(x) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} L(s,f)\frac{x^s}{s}\,\mathrm{d}s + O\bigg(N^{1/2}\bigg\{\bigg(\frac{x}{M}\bigg)^{1/2} + 1\bigg\}(Nx)^\varepsilon\bigg). \tag{3.5}$$

We shift the line of integration horizontally to $\Re e\, s = -\varepsilon$, the main term gives

$$\frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} L(s,f)\frac{x^s}{s}\,\mathrm{d}s = L(0,f) + \frac{1}{2\pi i}\int_{\mathscr{L}} L(s,f)\frac{x^s}{s}\,\mathrm{d}s, \tag{3.6}$$

where $\mathscr{L}$ is the contour joining the points $\kappa \pm iT$ and $-\varepsilon \pm iT$. Using the convexity bound

$$L(\sigma+it,f) \ll \big(\sqrt{N}(k+|t|)\big)^{\max\{0,1-\sigma\}+\varepsilon} \qquad (-\varepsilon \leqslant \sigma \leqslant \kappa),$$

the integrals over the horizontal segments and the term $L(0,f)$ can be absorbed in $O\big((NTx)^\varepsilon(N^{1/2}+T^{-1}x)\big)$. The O-constant depends on k and ε, and in the sequel, such a dependence in implied constants will be tacitly allowed.

To handle the integral over the vertical segment $\mathscr{L}_{\mathrm{v}} := [-\varepsilon - iT, -\varepsilon + iT]$, we invoke the functional equation

$$\bigg(\frac{\sqrt{N}}{2\pi}\bigg)^s \Gamma\bigg(s + \frac{k-1}{2}\bigg)L(s,f) = i^k\eta_f\bigg(\frac{\sqrt{N}}{2\pi}\bigg)^{1-s}\Gamma\bigg(1-s+\frac{k-1}{2}\bigg)L(1-s,f)$$

where $\eta_f := \mu(N)\lambda_f(N)\sqrt{N} \in \{\pm 1\}$ (see [10, p.375] with an obvious change of notation). Then we deduce that

$$\frac{1}{2\pi i}\int_{\mathscr{L}_{\mathrm{v}}} L(s,f)\frac{x^s}{s}\,\mathrm{d}s = i^k\eta_f\sum_{n\geqslant 1}\frac{\lambda_f(n)}{n}I_{\mathscr{L}_{\mathrm{v}}}(nx), \tag{3.7}$$

where

$$I_{\mathscr{L}_{\mathrm{v}}}(y) := \frac{1}{2\pi i}\int_{\mathscr{L}_{\mathrm{v}}}\bigg(\frac{4\pi^2}{N}\bigg)^{s-1/2}\frac{\Gamma(1-s+(k-1)/2)}{\Gamma(s+(k-1)/2)}\frac{y^s}{s}\,\mathrm{d}s.$$

The quotient of the two gamma factors is

$$|t|^{1-2\sigma}e^{-2i(t\log|t|-t)+i\mathrm{sgn}(t)\pi(k-1)/2}\{1+O(t^{-1})\}$$

for bounded σ and any $|t| \geqslant 1$, where the implied constant depends on σ and k. Together with the second mean value theorem for integrals (see [20], Theorem I.0.3), we obtain

$$\begin{aligned} I_{\mathscr{L}_{\mathrm{v}}}(nx) &\ll N^{1/2}\left(\frac{N}{nx}\right)^{\varepsilon}\left(\left|\int_1^T t^{2\varepsilon}e^{-ig(t)}\,\mathrm{d}t\right| + T^{2\varepsilon}\right) \\ &\ll N^{1/2}\left(\frac{NT^2}{nx}\right)^{\varepsilon}\left(\left|\int_a^b e^{-ig(t)}\,\mathrm{d}t\right| + 1\right) \end{aligned} \tag{3.8}$$

for some $1 \leqslant a \leqslant b \leqslant T$, where $g(t) := t\log\big(Nt^2/(4\pi^2 nx)\big) - 2t$. In view of (3.4), we have

$$g'(t) = -\log(4\pi^2 nx/(Nt^2)) < 0 \qquad \text{and} \qquad |g'(t)| \geqslant |\log(n/(M+\tfrac{1}{2}))|$$

for $n \geqslant M+1$ and $1 \leqslant t \leqslant T$. Using (1.1) and [20, Theorem I.6.2], we infer that

$$\begin{aligned} \sum_{n>M}\frac{\lambda_f(n)}{n}I_{\mathscr{L}_{\mathrm{v}}}(nx) &\ll N^{1/2}\left(\frac{NT^2}{x}\right)^{\varepsilon}\sum_{n>M}\frac{d(n)}{n^{1+\varepsilon}}\left(\left|\log\frac{n}{M+\frac{1}{2}}\right|^{-1} + 1\right) \\ &\ll N^{1/2}\left(\frac{NT^2}{x}\right)^{\varepsilon}\bigg\{\sum_{M<n\leqslant 2M}\frac{d(n)(M+\frac{1}{2})}{n^{1+\varepsilon}|n-M-\frac{1}{2}|} + \frac{1}{M^{\varepsilon/2}}\bigg\} \\ &\ll N^{1/2}\left(\frac{NT^2}{\sqrt{M}x}\right)^{\varepsilon} \\ &\ll N^{1/2}(Nx)^{\varepsilon}. \end{aligned} \tag{3.9}$$

For $n \leqslant M$, we extend the segment of integration $\mathscr{L}_{\mathrm{v}}$ to an infinite line $\mathscr{L}_{\mathrm{v}}^*$ in order to apply Lemma 1 in [3]. Write

$$\mathscr{L}_{\mathrm{v}}^{\pm} := [\tfrac{1}{2}+\varepsilon \pm iT, \tfrac{1}{2}+\varepsilon \pm i\infty), \qquad \mathscr{L}_{\mathrm{h}}^{\pm} := [-\varepsilon \pm iT, \tfrac{1}{2}+\varepsilon \pm iT]$$

and define $\mathscr{L}_{\mathrm{v}}^*$ to be the positively oriented contour consisting of $\mathscr{L}_{\mathrm{v}}$, $\mathscr{L}_{\mathrm{v}}^{\pm}$ and $\mathscr{L}_{\mathrm{h}}^{\pm}$. The contribution over the horizontal segments $\mathscr{L}_{\mathrm{h}}^{\pm}$ is

$$\begin{aligned} I_{\mathscr{L}_{\mathrm{h}}^{\pm}}(nx) &\ll \int_{-\varepsilon}^{1/2-\varepsilon}\left(\frac{4\pi^2}{N}\right)^{\sigma-1/2}T^{1-2\sigma}\frac{(nx)^{\sigma}}{T}\,\mathrm{d}\sigma \\ &\ll N^{1/2}\int_{-\varepsilon}^{1/2-\varepsilon}\left(\frac{nx}{NT^2}\right)^{\sigma}\mathrm{d}\sigma \\ &\ll N^{1/2}(Nx)^{\varepsilon}. \end{aligned}$$

As in (3.8), for $n \leqslant M$ we get that

$$\begin{aligned} I_{\mathscr{L}_{\mathrm{v}}^{\pm}}(nx) &\ll N^{1/2}\left(\frac{nx}{N}\right)^{1/2+\varepsilon}\left(\int_T^{\infty} t^{-1-2\varepsilon}e^{-ig(t)}\,\mathrm{d}t + \frac{1}{T^{1+2\varepsilon}}\right) \\ &\ll N^{1/2}\left(\frac{nx}{NT^2}\right)^{1/2+\varepsilon}\left(\left|\log\frac{M+\frac{1}{2}}{n}\right|^{-1} + 1\right) \\ &\ll N^{1/2}\left(\left|\log\frac{M+\frac{1}{2}}{n}\right|^{-1} + 1\right). \end{aligned}$$

So

$$\begin{aligned}\sum_{n\leqslant M}\frac{\lambda_f(n)}{n}\big(I_{\mathscr{L}_{\mathrm{v}}^{\pm}}(nx)+I_{\mathscr{L}_{\mathrm{h}}^{\pm}}(nx)\big) &\ll \sum_{n\leqslant M}\frac{d(n)}{n}\big(\big|I_{\mathscr{L}_{\mathrm{v}}^{\pm}}(nx)\big|+\big|I_{\mathscr{L}_{\mathrm{h}}^{\pm}}(nx)\big|\big)\\ &\ll N^{1/2}(Nx)^{\varepsilon}.\end{aligned}\tag{3.10}$$

Now all the poles of the integrand in

$$I_{\mathscr{L}_{\mathrm{v}}^{*}}(y)=\frac{\sqrt{N}}{2\pi}\frac{1}{2\pi i}\int_{\mathscr{L}_{\mathrm{v}}^{*}}\frac{\Gamma(1-s+(k-1)/2)\Gamma(s)}{\Gamma(s+(k-1)/2)\Gamma(1+s)}\left(\frac{4\pi^2 y}{N}\right)^{s}\mathrm{d}s$$

lie on the right of the contour $\mathscr{L}_{\mathrm{v}}^{*}$. After a change of variable s into $1-s$, we see that

$$I_{\mathscr{L}_{\mathrm{v}}^{*}}(y)=\frac{\sqrt{N}}{2\pi}I_0\left(\frac{4\pi^2 y}{N}\right),$$

with

$$I_0(t):=\frac{1}{2\pi i}\int_{\mathscr{L}_{\varepsilon}}\frac{\Gamma(s+(k-1)/2)\Gamma(1-s)}{\Gamma(1-s+(k-1)/2)\Gamma(2-s)}t^{1-s}\,\mathrm{d}s.$$

Here $\mathscr{L}_{\varepsilon}$ consists of the line $s=\frac{1}{2}-\varepsilon+i\tau$ with $|\tau|\geqslant T$, together with three sides of the rectangle whose vertices are $\frac{1}{2}-\varepsilon-iT$, $1+\varepsilon-iT$, $1+\varepsilon-iT$ and $\frac{1}{2}-\varepsilon+iT$. Clearly our I_0 is a particular case of I_ρ defined in [3, Lemma 1], corresponding to the choice of parameters $\rho=0$, $\delta=A=1$, $\omega=1$, $h=2$, $k_0=-(2k+1)/4$. It hence follows that

$$I_{\mathscr{L}_{\mathrm{v}}^{*}}(nx)=\frac{i^k(nNx)^{1/4}}{\pi\sqrt{2}}\cos\left(4\pi\sqrt{\frac{nx}{N}}-\frac{\pi}{4}\right)+O\left(\frac{N^{3/4+\varepsilon}}{(nx)^{1/4}}\right),\tag{3.11}$$

The value of e_0' in Lemma 1 of [3] is $1/\sqrt{\pi}$ by direct computation. We conclude

$$\begin{aligned}\sum_{n\leqslant M}\frac{\lambda_f(n)}{n}I_{\mathscr{L}_{\mathrm{v}}}(nx)=&\frac{i^k(Nx)^{1/4}}{\pi\sqrt{2}}\sum_{n\leqslant M}\frac{\lambda_f(n)}{n^{3/4}}\cos\left(4\pi\sqrt{\frac{nx}{N}}-\frac{\pi}{4}\right)\\ &+O\left(N^{1/2}\left\{\left(\frac{N}{x}\right)^{1/4}+1\right\}(Nx)^{\varepsilon}\right),\end{aligned}\tag{3.12}$$

from (3.10) and (3.11), and finally the asymptotic formula (3.2) by (3.5)-(3.7), (3.9) and (3.12). □

Following Theorem 1 of [8], we have the next lemma.

Lemma 3.2. *Let $f\in\mathrm{H}_k^{*}(N)$. There exist positive absolute constants C, c_1, c_2 such that for all sufficiently large $X\geqslant N^2X_0(k)$, we can find $x_1, x_2\in[X, X+C_NX^{1/2}]$ for which*

$$S_f^{*}(x_1)>c_1(NX)^{1/4}\qquad\text{and}\qquad S_f^{*}(x_2)<-c_2(NX)^{1/4},$$

where $C_N:=CN^{1/2}\Psi(N)^3$ and $X_0(k)$ is a constant depending only on k. The same result also holds for $S_f(x)$.

Proof. Define

$$K_\tau(u) := (1-|u|)(1+\tau\cos(4\pi\alpha u)),$$

where $\tau = 1$ or -1 and α is a (large) parameter, both chosen at our disposal. Consider the following integral

$$r_\beta = r_\beta(\alpha,\tau,t) := \int_{-1}^{1} K_\tau(u)\cos\left(4\pi(t+\alpha u)\sqrt{\beta}-\frac{\pi}{4}\right)\mathrm{d}u,$$

where $t \in \mathbb{N}$ and $\beta > 0$. Because

$$w(\xi) := \int_{-1}^{1}(1-|u|)e^{i2\pi\xi u}\,\mathrm{d}u = \left(\frac{\sin\pi\xi}{\pi\xi}\right)^2 = \begin{cases} 1 & \text{if } \xi = 0,\\ O\big(\min(1,\xi^{-2})\big) & \text{if } \xi \neq 0,\end{cases}$$

we can write, with the notation $\alpha_\beta := 2\alpha\sqrt{\beta}$ and $\alpha_\beta^\pm := 2\alpha(\sqrt{\beta}\pm 1)$,

$$\begin{aligned} r_\beta &= \int_{-1}^{1}(1-|u|)\left(1+\tau\frac{e^{i4\pi\alpha u}+e^{-i4\pi\alpha u}}{2}\right)\Re e\, e^{i\{4\pi(t+\alpha u)\sqrt{\beta}-\pi/4\}}\,\mathrm{d}u\\ &= \Re e\, e^{i(4\pi t\sqrt{\beta}-\pi/4)}\int_{-1}^{1}(1-|u|)\left(e^{i2\pi\alpha_\beta u}+\frac{\tau}{2}e^{i2\pi\alpha_\beta^+ u}+\frac{\tau}{2}e^{i2\pi\alpha_\beta^- u}\right)\mathrm{d}u\\ &= \left(w(\alpha_\beta)+\frac{\tau}{2}w(\alpha_\beta^+)+\frac{\tau}{2}w(\alpha_\beta^-)\right)\cos\left(4\pi t\sqrt{\beta}-\frac{\pi}{4}\right)\\ &= \delta_{\beta=1}\frac{\tau}{2\sqrt{2}}+O\bigg(\min\left(1,\frac{1}{\alpha^2\beta}\right)+\delta_{\beta\neq1}\min\left(1,\frac{1}{(\alpha_\beta^-)^2}\right)\bigg), \end{aligned} \tag{3.13}$$

where the O-constant is absolute,

$$\delta_{\beta=1} := \begin{cases} 1 & \text{if } \beta = 1\\ 0 & \text{otherwise}\end{cases} \qquad \text{and} \qquad \delta_{\beta\neq1} := 1-\delta_{\beta=1}.$$

The last error term in (3.13) appears only when $\beta \neq 1$.

For all $X \geqslant N^2X_0(k)$ (whose value will be specified below), we write $T = (X/N)^{1/2}$ and $t = [T]+1 \in \mathbb{N}$, and consider the convolution

$$J_\tau = \int_{-1}^{1} F_f(t+\alpha u)K_\tau(u)\,\mathrm{d}u,$$

where

$$F_f(t+\alpha u) := \frac{\pi\sqrt{2}}{\eta_f}\frac{S_f^*(N(t+\alpha u)^2)}{\sqrt{N(t+\alpha u)}}.$$

By Lemma 3.1 with $M = NT^2 = X$, we deduce that

$$F_f(t+\alpha u) = \sum_{d|N}\frac{(-1)^{\omega(d)}\lambda_f(d)}{d^{1/4}}\sum_{n\leqslant M}\frac{\lambda_f(n)}{n^{3/4}}\cos\left(4\pi(t+\alpha u)\sqrt{\frac{n}{d}}-\frac{\pi}{4}\right)+O_k\bigg(\frac{1}{T^{1/4}}\bigg),$$

and

$$J_\tau = \sum_{d|N}\frac{(-1)^{\omega(d)}\lambda_f(d)}{d^{1/4}}\sum_{n\leqslant M}\frac{\lambda_f(n)}{n^{3/4}}r_{n/d}+O_k\bigg(\frac{1}{T^{1/4}}\bigg) \tag{3.14}$$

by (1.2).

Next we estimate the contribution of the O-term in (3.13) to J_τ. Using (1.2) and (1.1) again, its contribution to J_τ is

$$\ll \sum_{d|N} \frac{1}{d^{3/4}} \bigg\{ \sum_{n\leqslant M} \frac{d(n)}{n^{3/4}} R'_{d,n}(\alpha) + \sum_{\substack{n\leqslant M\\ n\neq d}} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \bigg\}, \tag{3.15}$$

where

$$R'_{d,n}(\alpha) := \min\bigg(1, \frac{d}{\alpha^2 n}\bigg), \qquad R''_{d,n}(\alpha) := \min\bigg(1, \frac{d}{\alpha^2|\sqrt{n}-\sqrt{d}|^2}\bigg).$$

Consider the second sum in the curly braces. We separate n into

$$n \leqslant \alpha_- d, \qquad \alpha_- d < n < \alpha_+ d \qquad \text{or} \qquad \alpha_+ d \leqslant n$$

where $\alpha_\pm := (1-\alpha^{-1/2})^{\mp 2}$, and $R''_{d,n}(\alpha)$ is $\leqslant 1/\alpha$, 1 or $d/(\alpha n)$ accordingly. Therefore,

$$\sum_{\substack{n\leqslant M\\ n\neq d}} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \leqslant \frac{1}{\alpha} \sum_{n\leqslant \alpha_- d} \frac{d(n)}{n^{3/4}} + \sum_{\substack{\alpha_- d<n<\alpha_+ d\\ n\neq d}} \frac{d(n)}{n^{3/4}} + \frac{d}{\alpha} \sum_{n>\alpha_+ d} \frac{d(n)}{n^{7/4}}.$$

Obviously the first and last terms on the right-hand side are $\ll \alpha^{-1} d^{1/4} \log(2d)$. Note that $n \asymp d$ in the second sum. So, by using Shiu's Theorem 2 in [19] it follows

$$\begin{aligned} \sum_{\substack{\alpha_- d<n<\alpha_+ d\\ n\neq d}} \frac{d(n)}{n^{3/4}} &\ll d^{-3/4} \sum_{\substack{\alpha_- d<n<\alpha_+ d\\ n\neq d}} d(n) \\ &\ll \alpha^{-1/2} d^{1/4} \log(2d) \end{aligned}$$

if $d > \alpha$. Otherwise (i.e. $d \leqslant \alpha$), pulling out $d(n) \ll n^\varepsilon \ll d^\varepsilon \ll \alpha^\varepsilon$, we have

$$\begin{aligned} \sum_{\substack{\alpha_- d<n<\alpha_+ d\\ n\neq d}} d(n) n^{-3/4} &\ll \alpha^\varepsilon d^{-3/4} \sum_{\substack{\alpha_- d<n<\alpha_+ d\\ n\neq d}} 1 \\ &\ll \alpha^\varepsilon d^{-3/4} \alpha^{-1/2} d \\ &\ll \alpha^{-1/3} d^{1/4} \log(2d). \end{aligned}$$

(We can assume that $(\alpha_+ - \alpha_-)d \geqslant \alpha^{-1/2} d \geqslant c'$ for a small constant c', otherwise the last sum is empty.) Hence

$$\sum_{\substack{n\leqslant M\\ n\neq d}} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \ll \alpha^{-1/3} d^{1/4} \log(2d).$$

The first sum in the bracket of (3.15) can be treated in the same fashion (even more easily). Thus, (3.15) is bound by

$$\ll \alpha^{-1/3} \sum_{d|N} \frac{\log(2d)}{d^{1/2}} =: \alpha^{-1/3} \Psi(N).$$

We conclude from (3.14) with (3.13) and (1.2) that

$$J_\tau = \frac{\tau}{2\sqrt{2}}\sum_{d|N}\frac{(-1)^{\omega(d)}}{d^2} + O\bigg(\frac{\Psi(N)}{\alpha^{1/3}}\bigg) + O_k\bigg(\frac{1}{T^{1/4}}\bigg),$$

where the implied constant is absolute in the first O-term, but depends on k in the second. Noticing that

$$\sum_{d|N}\frac{(-1)^{\omega(d)}}{d^2} = \prod_{p|N}\bigg(1-\frac{1}{p^2}\bigg) \geqslant \frac{6}{\pi^2}$$

and $T \geqslant \sqrt{NX_0(k)}$, we take $\alpha = C\Psi(N)^3$ with a large absolute constant C and a large $X_0(k)$ so that both O-terms $O(\alpha^{-1/3}\Psi(N))$ and $O_k(T^{-1/4})$ are $\leqslant \cos(\pi/4)/\pi^2 = 1/(\pi^2\sqrt{2})$. Therefore

$$J_{-1} < -1/(\pi^2\sqrt{2}) \qquad \text{and} \qquad J_1 > 1/(\pi^2\sqrt{2}).$$

With the nonnegativity of $K_\tau(u)$ and the estimate

$$1-(2\pi\alpha)^{-2} \leqslant \int_{-1}^{1} K_\tau(u)\,\mathrm{d}u \leqslant 2 \qquad (\tau = \pm 1),$$

we have

$$2F_f(t+\alpha\eta_+) \geqslant 1/(\pi^2\sqrt{2}) \quad \text{and} \quad \big(1-(2\pi\alpha)^{-2}\big)F_f(t+\alpha\eta_-) \leqslant -1/(\pi^2\sqrt{2})$$

for some $\eta_+, \eta_- \in [-1,1]$. Let $C_N = CN^{1/2}\Psi(N)^3$. As

$$X - 3C_N\sqrt{X} \leqslant N(t+\alpha\eta_\pm)^2 \leqslant X + 3C_N\sqrt{X},$$

our assertion follows from the definition of F_f and replacing $X - 3C_N\sqrt{X}$ by X. □

Now we are ready to prove Theorem 2.

We exploit the consecutive sign changes of $S_f^*(x)$. Let $x \geqslant N^2X_0(k)$ where $X_0(k)$ takes the value as in Lemma 3.2. We apply Lemma 3.2 to the intervals $[x, x+C_Nx^{1/2}]$ and $[y, y+C_Ny^{1/2}]$ where $y = x+C_Nx^{1/2}$. Over each of the intervals, $S_f^*(x)$ attains in magnitude $(Nx)^{1/4}$ in both positive and negative directions. Hence, we can find three points $x < x_1 < x_2 < x_3 < x+3C_Nx^{1/2}$ such that $S_f^*(x_i)$ $(i=1,2,3)$ takes alternate signs and their absolute values are $\gg (Nx)^{1/4}$. (Note that $2\sqrt{x} \geqslant \sqrt{x+C_N\sqrt{x}}$.) It follows that the two differences

$$S_f^*(x_2) - S_f^*(x_1) = \sum_{\substack{x_1<n\leqslant x_2\\ (n,N)=1}} \lambda_f(n)$$

and

$$S_f^*(x_3) - S_f^*(x_2) = \sum_{\substack{x_2<n\leqslant x_3\\ (n,N)=1}} \lambda_f(n)$$

have absolute values $\gg (Nx)^{1/4}$ but are of opposite signs. This implies (1.6), since for example, if

$$\sum_{\substack{a<n<b\\ (n,N)=1}} \lambda_f(n) < -c'(Nx)^{1/4}$$

for some constant $c' > 0$ and $b \ll x$, then we have

$$\begin{aligned} c'(Nx)^{1/4} &< \sum_{\substack{a<n<b,\,(n,N)=1\\ \lambda_f(n)<0}} \big(-\lambda_f(n)\big) \\ &\ll x^{\varepsilon} \sum_{\substack{a<n<b,\,(n,N)=1\\ \lambda_f(n)<0}} 1. \end{aligned}$$

This completes the proof of Theorem 2. □

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